Group Theory 1

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Permutation:- In a set X (non-empty) a permutation is a bijection $f: X \to X$

Set of all such permutations is called S_X.

$|S_n| = n!$ for $|X| = n$

$f: X \to X$, $j \to f(j)$

$j \in \{1, 2, \ldots, n\}$
$f(j) \in \{1, 2, \ldots, n\}$

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad f_1 f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$f_1(f_2(1)) = f_1(2) = 2$
$f_1(f_2(2)) = f_1(3) = 1$
$f_1(f_2(3)) = f_1(1) = 3$

$$f_2 f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$f_1 f_2 \neq f_2 f_1$

Identity $1_X : X \to X$, $j \to 1_X(j)$ $\qquad 1_X f = f 1_X = f \quad \forall f \in S_X$

Q) For each $\alpha \in S_X$, prove that there is $\beta \in S_X$ such that $\alpha\beta = 1 = \beta\alpha$

Ans:- $X = \{x_1, x_2, \ldots, x_n\}$

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \alpha(x_1) & \alpha(x_2) & \cdots & \alpha(x_n) \end{pmatrix}$$

α is bijection so α^{-1} exists. Let $\beta = \alpha^{-1}$

Then $$\beta = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \alpha^{-1}(x_1) & \alpha^{-1}(x_2) & \cdots & \alpha^{-1}(x_n) \end{pmatrix}$$

$$\alpha\beta = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = 1_X$$

$$\beta\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = 1_X$$

$\alpha(\beta(x_i)) = \alpha(\alpha^{-1}(x_i)) = x_1$

$\alpha\beta = \beta\alpha = 1$

Q. $\alpha, \beta, \gamma \in S_X$, prove that, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

Ans:- $$\beta\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \beta\gamma(x_1) & \beta\gamma(x_2) & \cdots & \beta\gamma(x_n) \end{pmatrix}$$

$$\alpha(\beta\gamma) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \alpha(\beta\gamma)(x_1) & \alpha(\beta\gamma)x_2 & \cdots & \alpha(\beta\gamma)(x_n) \end{pmatrix}$$

$f: X \to Y$
$g: Y \to Z$
$h: Z \to A$

$h(gf) = (hg)f$

f, g are equal if only if $f(x) = g(x) \ \forall x \in$ domain

$$\alpha(\beta\gamma) = \begin{pmatrix} x_1 & \cdots & x_n \\ \alpha(\beta(\gamma(x_1))) & \cdots & \alpha(\beta(\gamma(x_n))) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & \cdots & x_n \\ (\alpha\beta)\gamma(x_1) & \cdots & (\alpha\beta)\gamma(x_n) \end{pmatrix} = (\alpha\beta)\gamma$$

$$= \begin{pmatrix} x_1 & \cdots\cdots & x_n \\ (\alpha\beta)(\gamma(x_1)) & \cdots & (\alpha\beta)\gamma(x_n) \end{pmatrix} = (\alpha\beta)\gamma$$

Cycles:- If $x \in X$ and $f \in S_X$, then f fixes x if $f(x) = x$ and f moves x if $f(x) \neq x$

Example:- $f(i_1) = i_2,\ f(i_2) = i_3,\ \ldots,\ f(i_{r-1}) = i_r,\ f(i_r) = i_1$

$f(i_k) = i_k$ if $k > r$

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_r & i_{r+1} & \cdots & i_n \\ i_2 & i_3 & & i_1 & i_{r+1} & \cdots & i_n \end{pmatrix}$$ $i_k \in \{1, \ldots, n\}$ and distinct.

r-cycle

2-cycle $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

Total no of arrangements in r-cycle $= r$

$\alpha(\alpha(i_1)) = \alpha(i_2) = i_3$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\ 2\ 3\ 4)$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix} = (1\ 2\ 3\ 4)(5) = (1\ 2\ 3\ 4)$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix} = (1\ 5\ 3\ 4\ 2)$

1-row notation for permutation in cycles

$f_1 = (1\ 2)$

$f_2 = (1\ 3\ 4\ 2\ 5)$

$f_1 f_2 = (1\ 2)(1\ 3\ 4\ 2\ 5)$

$= (1\ 3\ 4)(2\ 5)$

•> Two permutations f_1 and $f_2 \in S_X$ are disjoint if for every $x \in X$ it is moved by one then it is fixed by the other

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Q> Prove that $(1\ 2\ \cdots\ n{-}1\ n) = (2\ 3\ \cdots\ n\ 1)$
$= (3\ 4\ \cdots\ 1\ 2) = \cdots = (n\ 1\ \cdots\ n{-}1)$.

$= (3\ 4\ \cdots\ 1\ 2)$

Q) If $1 \le r \le n$, then we have that there are $\frac{1}{r}[n(n-1)\cdots(n-r+1)]$ r-cycles in S_n.

Q) Prove if $\alpha, \beta, \gamma \in S_X$ and we have either $\alpha\beta = \alpha\gamma$ or $\beta\alpha = \gamma\alpha$ then $\beta = \gamma$.

Ans:- Idea:- $\alpha\beta \neq \beta\alpha$

$\alpha\gamma \neq \gamma\alpha$

$\alpha\beta = \alpha\gamma \longrightarrow \beta = \gamma$

$\beta\alpha = \gamma\alpha \longrightarrow \gamma = \beta$

Proof:- either $\alpha^{-1}\alpha\beta = \alpha^{-1}\alpha\gamma \Rightarrow \beta = \gamma$

or $\beta\alpha\alpha^{-1} = \gamma\alpha\alpha^{-1} \Rightarrow \beta = \gamma$